# Homework #4

due 10/25/11, in class

**Problem 1** Consider the 4 x 5 matrix $A$ given below. Consider the problem of finding a path which starts at $A_{11}$ (upper left corner) and ends at $A_{45}$ (lower right corner) which moves only to the right and down.

$$A = \begin{bmatrix} 4 & 9 & 3 & 6 & 3 \\ 5 & 6 & 6 & 4 & 4 \\ 6 & 7 & 1 & 1 & 0 \\ 4 & 3 & 5 & 1 & 9 \end{bmatrix}$$

Suppose you want to maximize the sum of the entries encountered along the path. Set this up as a dynamic programming problem, and solve using backward induction.

Start at $A_{45}$. The value of the maximized path from that point forward is obviously 9. Now, consider $A_{35}$ and $A_{44}$. Clearly, $V[A_{44}] = 10$ and $V[A_{35}] = 9$. Now, $V[A_{34}] = 1 + \max\{V[A_{35}], V[A_{44}]\} = 11$. Continue working backwards in this manner. The following matrix gives the maximized value at each of the 20 possible states:

$$A = \begin{bmatrix} 44 & 40 & 26 & 23 & 16 \\ 36 & 31 & 23 & 17 & 13 \\ 31 & 25 & 16 & 11 & 9 \\ 22 & 18 & 15 & 10 & 9 \end{bmatrix}$$

So, the path with the maximized sum is $A_{11}, A_{12}, A_{22}, A_{32}, A_{42}, A_{43}, A_{44}, A_{45}$, and the maximized sum is 44.

**Problem 2** Consider the following dynamic optimization problem:[1]

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \quad \text{subject to} \quad c_t + k_{t+1} = Ak_t^{\alpha}$$

$$k_0 \text{ given}$$

**a.** Write down the Bellman equation associated with this problem.

In time period $t$, the state variable is $k_t$, and the control variables are $c_t$ and $k_{t+1}$. The Bellman equation is:

$$V[k] = \max_{k'} \log(Ak^{\alpha} - k') + \beta V[k'] \tag{1}$$

Note there are 2 other versions of (1), one with $c$ as the control variable, and one with both $c$ and $k'$ as control variables, and a constraint added.

**b.** Truncate the infinite horizon problem at $T$. Solve for the optimal period 0 policy rule $k_1(k_0)$, as a function of $T$ (do this by solving for $k_1(k_0)$ under $T = 1$, $T = 2$, etc, and looking for a pattern).

$$k_1(k_0) = \frac{\alpha\beta + (\alpha\beta)^2 + ... + (\alpha\beta)^T}{1 + \alpha\beta + (\alpha\beta)^2 + ... + (\alpha\beta)^T} Ak_0^{\alpha} \tag{2}$$

[1] For those of you who have taken a macro course, this is the planner's version of the Ramsey growth model with full depreciation and specific functional forms assumed for $u$ and $f$.

**c.** Show that the limit as $T \to \infty$ of $k_1(k_0)$ is equal to the infinite-horizon policy rule $k'(k)$ derived under guess and verify. Noting that $\sum_{t=1}^{\infty}(\alpha\beta)^t = \frac{1}{1-\alpha\beta}$, we get

$$\begin{aligned}\lim_{T\to\infty} k_1(k_0) &= \frac{\frac{\alpha\beta}{1-\alpha\beta}}{\frac{1}{1-\alpha\beta}} A k_0^{\alpha} \\ &= \alpha\beta A k_0^{\alpha} \qquad (3)\end{aligned}$$

**d.** Do you think it's true or false that $\lim_{T\to\infty} k_2(k_1) = \lim_{T\to\infty} k_1(k_0)$? Why?

This is true. It can be verified by direct calculation, or by noting that as the problem extends to an infinite horizon, the time period becomes irrelevant in calculating the value function and policy rule.

**Problem 3** Consider the fish hatchery problem discussed in class and as problem 11.2 in the Sundaram book.[2] Use the optimal policy rule derived in class, meaning that the hatchery optimally harvests $y$ fish if it begins period $T - k$ with $x$ fish, where:

$$y = \frac{x}{1 + \alpha + \alpha^2 + ... + \alpha^k} \tag{4}$$

Derive numerical answers to each of the following:

**a.** Suppose $T = 5$, $\alpha = .9$, and the stock of fish at the beginning of period 0 is 1,000. How many fish does the hatchery harvest in periods 0, 1, 2, 3, 4, and 5, and what is the total profit earned by the hatchery?

**b.** Now suppose that instead of harvesting fish according to (4), the hatchery harvests 50% of all its fish each period. Now what are its total profits, beginning in period 0?

**c.** Go back to part a. Approximately how many fish would the hatchery have to have in stock at the beginning of period 0 for their total profit, from that point forward, to exceed \$15?

**d.** Go back to part a. Would the hatchery's total profits increase or decrease were Armageddon to be postponed by one period (so now $T = 6$)? If you answered "increase", how can this be, given that their total initial number of fish has not changed, and given that so many fish die each period?

**e.** Plot the relationship between $T$ and total profits (this will be a very rough plot... just pick a few values of $T$, calculate profits for each, and see if you can determine if the relationship is increasing or decreasing, concave or convex, over the values you picked).

**f.** Suppose $T = 5$, but the hatchery needs to decide between different water treatment policies, which will affect $\alpha$. Plot out the relationship between total profits and $\alpha$, in enough detail that the hatchery can make an educated guess about the marginal benefit of increasing $\alpha$.

[2]Note that the class notes and the Sundaram problem use slightly different notation.